CUT groups

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Group rings

Definition

G group, A ring. The *group ring* is

$$A[G] = \bigoplus_{g \in G} Ag$$

with the product

$$ag \cdot bh = (ab)(gh)$$
$$\forall a, b \in A; g, h \in G$$

CUT groups

- $\pm G = \{$*trivial units* of $\mathbb{Z}[G]\}$.
- $\pm Z(G) = \{$trivial central units of $\mathbb{Z}[G]\}$.

$$\pm Z(G) \subseteq Z(\mathcal{U}(\mathbb{Z}[G])) = \mathcal{U}(Z(\mathbb{Z}[G])).$$

Definition

G is *CUT (Central Units are Trivial)* if $Z(\mathcal{U}(\mathbb{Z}[G])) = \pm Z(G)$.

Objective: characterise CUT groups.

Characterisations

Theorem

G finite group. The following statements are equivalent:

1. G *is CUT.*
2. $Z(\mathcal{U}(\mathbb{Z}[G]))$ *is finite.*
3. *For each* $\chi \in Irr(G)$, $\mathbb{Q}(\chi)$ *is contained in an imaginary quadratic field.*
4. *For each* $g \in G$ *and* $j \in \mathbb{N}$ *coprime with* $|G|$, $g^j \sim g$ *or* $g^j \sim g^{-1}$.
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$(1) \implies (2)$ is obvious.

$(1) \impliedby (2)$ based on Berman-Higman's theorem.

Group representations and characters

Definition

F field, G group. An *F-representation of G* is a group homomorphism $\rho : G \to GL_n(F)$.
The character afforded by ρ is

$$\begin{aligned} \chi : G &\longrightarrow F \\ g &\longmapsto \chi(g) = tr(\rho(g)). \end{aligned}$$

ρ and χ can be linearly extended to $F[G]$.

Extended ρ is an *algebra homomorphism*.

Regular character

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- $\rho(g) := \mathcal{M}_{\mathcal{B}}(\lambda(g))$ is a *regular F-representation*.
- ρ affords the *regular character* ϕ.
- $$\forall g \in G \quad \phi(g) = \begin{cases} 0 & \text{if } g \neq 1; \\ |G| & \text{if } g = 1. \end{cases}$$

- $\alpha = \sum_{g \in G} a_g g \in \mathbb{Z}[G] \subseteq \mathbb{C}[G]$, $a_1 \neq 0$ and $\alpha^m = 1$ for some $m \in \mathbb{Z}$.

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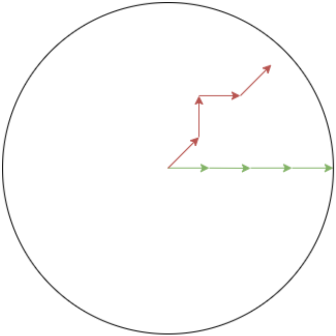

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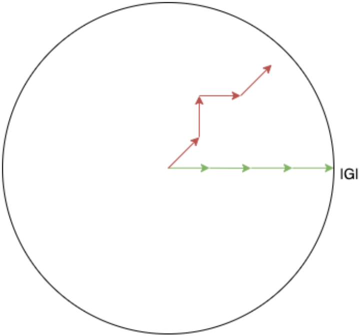

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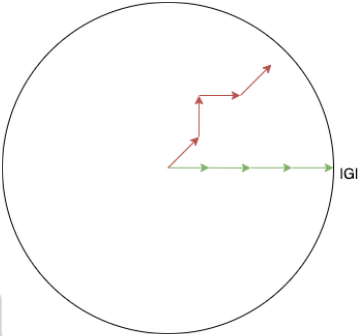

Theorem (Berman-Higman's theorem)

G finite, $\alpha = \sum_{g \in G} a_g g \in \mathbb{Z}[G]$, $a_1 \neq 0$ and α of finite order $\implies \alpha = \pm 1$.

$\alpha = \sum_{g \in G} a_g g \in Z(\mathbb{Z}[G])$, $a_h \neq 0$ and $\alpha^m = 1$ for some $m \in \mathbb{Z}$.

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Corollary

$0 \neq \alpha \in Z(\mathbb{Z}[G])$, α *of finite order* $\implies \alpha \in \pm Z(G)$.

Theorem

G finite group. The following statements are equivalent:

1. *G is CUT.*
2. *$Z(\mathcal{U}(\mathbb{Z}[G]))$ is finite.*
3. *For each $\chi \in Irr(G)$, $\mathbb{Q}(\chi)$ is contained in an imaginary quadratic field.*
4. *For each $g \in G$ and $j \in \mathbb{N}$ coprime with $|G|$, $g^j \sim g$ or $g^j \sim g^{-1}$.*
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For each $\chi \in Irr(G)$ $\exists ! f_i : \chi(f_i) \neq 0$, denoted by $e_{\mathbb{Q}}(\chi)$.

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Theorem

$\chi \in Irr(G)$, then $Z(e_{\mathbb{Q}}(\chi)\mathbb{Q}[G])$ is $\mathbb{Q}$-isomorphic to

$$\mathbb{Q}(\chi) := \mathbb{Q}(\chi(g) | g \in G).$$

Dirichlet's Unit Theorem

Theorem (Dirichlet's Unit Theorem)

- *K number field,*
- *$\mathbb{A}_K$ its ring of integers,*
- *r real and $2s$ complex homomorphisms $K \to \mathbb{C}$.*

$$\mathcal{U}(\mathbb{A}_K) \cong W \times \mathbb{Z}^{r+s-1},$$

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where W is finite.

Corollary

$\mathcal{U}(\mathbb{A}_K)$ *finite* $\iff K \subseteq F : F$ *is a quadratic imaginary field.*

$\chi \in Irr(G), f = e_{\mathbb{Q}}(\chi)$

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\begin{array}{ccccc}
 & & & & Z(f\mathbb{Q}[G]) \\
 & & & & \updownarrow \\
\mathcal{U}(\mathbb{A}_{\mathbb{Q}(\chi)}) & \hookrightarrow & \mathbb{A}_{\mathbb{Q}(\chi)} & \hookrightarrow & \mathbb{Q}(\chi)
\end{array}
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Definition

A $\mathbb{Q}$-algebra, $\mathcal{O}$ is an *order* in A iff $\mathcal{O}$ is a subring of A, its additive group is finitely generated and $\mathcal{O}$ contains a $\mathbb{Q}$-base of A.

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Theorem

A is a $\mathbb{Q}$-algebra, $\mathcal{O}, \mathcal{O}'$ orders in A, then:

- *$\mathcal{O} \cap \mathcal{O}'$ is an order in A;*
- $[\mathcal{U}(\mathcal{O}) : \mathcal{U}(\mathcal{O} \cap \mathcal{O}')] < \infty$.

Thus, $\mathcal{U}(\mathcal{O})$ is finite if and only if $\mathcal{U}(\mathcal{O}')$ is finite.

- $Z(\mathbb{Z}[G])$ is an order in $Z(\mathbb{Q}[G])$.

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(3) $\iff$ (4) proof by Ritter and Sehgal.

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By linear independence of $Irr(G)$

$$Gal(\mathbb{Q}(\chi)/\mathbb{Q}) \subseteq \{\text{identity}, \text{conjugation}\}.$$

$3 \implies 4.$

$$\begin{aligned} T(g) : Irr(G) &\longrightarrow \mathbb{C} \\ \chi &\longmapsto T(g)(\chi) = \chi(g). \end{aligned}$$

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Proof by Bächle.

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- $$\begin{aligned} \phi : Gal(\mathbb{Q}(\xi)/\mathbb{Q}) &\longrightarrow Aut(\langle g \rangle) \\ \sigma_j &\longmapsto \phi(\sigma_j) = \tau_j. \end{aligned}$$
- Restricts to

$$Gal(\mathbb{Q}(\xi)/\mathbb{Q}(g)) \leftrightarrow \{g \mapsto hgh^{-1} | h \in N_G(\langle g \rangle)\}.$$

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